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# Essential Self-adjointness of the Schrödinger Operator with Magnetic Vector Potential

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We prove essential self-adjointness of the operator under weak conditions on the coefficients.

## 1. INTRODUCTION

The operator we study is of the form

$$L = \sum_{i,k=1}^n a_{ik}(D_i + b_i(x))(D_k + b_k(x)) + q(x) \quad (1.1)$$

in  $E^n$ , where  $(a_{jk})$  is a positive definite symmetric matrix,  $q$  and the  $b_k$  are real-valued functions, and  $D_k = \partial/\partial x_k$ . We shall impose conditions on the coefficients so that  $L$  will map  $D = C_0^\infty$  (the test functions) into  $L^2 = L^2(E^n)$  and be essentially self-adjoint on it (i.e., that its closure on  $D$  be self-adjoint). Our hypotheses will be stated in terms of the classes  $M_{\alpha,p}$  defined as follows. For  $1 \leq p < \infty$ ,  $\alpha > 0$  and  $\delta > 0$ , put

$$\begin{aligned} M_{\alpha,p,\delta}(h) &= \sup_x \int_{|x-y|<\delta} |h(y)|^p |x-y|^{\alpha-n} dy, & \alpha < n, \\ &= \sup_x \int_{|x-y|<\delta} |h(y)|^p |\log |x-y||^{p-1} dy, & \alpha = n, \\ &= \sup_x \int_{|x-y|<\delta} |h(y)|^p dy, & \alpha > n, \\ M_{\alpha,p}(h) &= M_{\alpha,p,1}(h). \end{aligned}$$

We shall let  $M_{\alpha,p}$  denote the class of all functions  $h$  such that  $M_{\alpha,p}(h) < \infty$ . We shall say that  $h$  is in  $M_{\alpha,p}^{\text{loc}}$  if  $\varphi h$  is in  $M_{\alpha,p}$  for every  $\varphi \in D$ . (For discussions of these classes see [2].)

The study of essential self-adjointness of (1.1) has a long history (see [2] for a partial bibliography). Until recently it was usual to assume that the  $b_k$  were continuously differentiable and that  $q_+(x) = \max[q(x), 0]$  was in  $M_{\alpha,2}^{\text{loc}}$  for some  $\alpha < 4$  (cf. [1]). Recently, Schechter [2] showed that the continuous differentiability of the  $b_k$  can be relaxed considerably. He assumed that

$$b = \sum b_k^2, \quad e = \sum a_{jk} D_j b_k \quad (1.2)$$

are both in  $M_{\alpha,2}^{\text{loc}}$  as well as  $q$ . On the other hand, for the case  $b = 0$ , Simon [3] was able to show that  $q \geq 0$  and in  $L^2$  sufficed. Kato [4] then generalized this to the case when the  $b_k$  are continuously differentiable,  $q_+ \in L_{\text{loc}}^2$  and  $q_- = q - q_+$  is in  $M_{2,1}$  with  $M_{2,1,\delta}(q_-) \rightarrow 0$  as  $\delta \rightarrow 0$ . (Actually he allows a more general situation.) Simon [5] then weakened the conditions on  $b_k$  to  $b_k \in L_{\text{loc}}^p$  for some  $p > n$  and  $\geq 4$ . (Obviously Simon had not seen the work of the author in [2], since his stipulations are stronger.)

The purpose of the present paper is to weaken the assumptions on the  $b_k$  even beyond those of [2] while retaining the weak stipulations of Kato on  $q$  (actually we weaken those a bit as well). One of our results is

THEOREM 1.1. *Assume*

- (A) *Each  $b_k$  is in  $M_{1,1}^{\text{loc}}$ .*
- (B)  *$b, e$  and  $q$  are in  $L_{\text{loc}}^2$ .*
- (C) *There is a  $p$  such that  $1 \leq p \leq 2$  and*

$$M_{2p,p,\delta}(q_-) \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \quad (1.3)$$

- (D) *If  $p < 2$ , then there is an  $l$  such that  $e^{-l|x|} q_-(x)$  is in  $L^r$ , where  $1/r = 1/p - \frac{1}{2}$ .*

*Then the operator  $L$  is essentially self-adjoint on  $D$ .*

*Remarks.* (1) Kato [4] assumes that (C) holds with  $p = 1$ . He also makes an assumption slightly stronger than (D).

(2) Assumption (D) is annoying; it is not needed in the older theory, which assumed  $q \in M_{\alpha,2}^{\text{loc}}$  for some  $\alpha < 4$  (cf. [1]).

(3) The theorem holds if we assume that hypothesis (C) is satisfied for some  $p > n/2$ . However, this implies that (C) is satisfied for  $p = 1$  (cf. [2]). The same is true if one assumes that there is a  $p < \infty$  and an  $\alpha < 2p$  such that  $M_{\alpha,p,\delta}(q_-) \rightarrow 0$  as  $\delta \rightarrow 0$  in place of (1.3).

4. In Section 4 we shall give some variations of Theorem 1.1 in which hypothesis (C) is replaced by other assumptions.

Our method requires us to estimate the norm of the operator  $q_-(c^2 - \Delta)^{-1}$  on  $L^p$ . We give a generalization of this (Theorem 2.2) which extends some of our results of [2]. For  $1 \leq p \leq 2$  we show that  $q(c^2 - \Delta)^{-s/2}$  is a bounded operator on  $L^p$  if  $q \in M_{sp,p}$ . This and related results are presented in the next section. In Section 3 we extend a very useful inequality due to Kato [4] to the case when the  $b_k$  are not in  $C'$ . This was done by Simon [5] in a more restrictive case. In Section 4 we give the proof of Theorem 1.1 and other variations based on the results of Sections 2 and 3.

## 2. ESTIMATES IN $L^p$

In this section we shall generalize some of the estimates of [2]. This will allow us to use condition (C) of Theorem 1.1. We begin with some preliminary results.

**THEOREM 2.1.** *Suppose  $1 < p \leq 2$ , and put  $p' = p/(p-1)$ . Let  $T$  be an operator defined by*

$$Tu(x) = \int K_1(x, y) K_2(x, y) u(y) dy, \quad (2.1)$$

*and assume that*

$$J^p = \sup_x \int \left( \int |K_1(z, y) K_2(x, y)|^{p'} dy \right)^{p-1} dz \quad (2.2)$$

*is finite. Then  $T$  is a bounded operator on  $L^p$  with norm  $\leq J$ .*

*Proof.* Put  $\varphi(y)^p = \int |K_1(z, y)|^p dz$ . Then

$$\begin{aligned} |(Tu, v)| &\leq \left( \iint |K_1(x, y)|^p \varphi(y)^{-p} |u(y)|^p dx dy \right)^{1/p} \\ &\quad \times \left( \iint |K_2(x, y)|^{p'} \varphi(y)^{p'} |v(x)|^{p'} dx dy \right)^{1/p'}. \end{aligned} \quad (2.3)$$

Now by Jessen's inequality (see [6, p. 148]),

$$\begin{aligned} \int |K_2(x, y)|^{p'} \varphi(y)^{p'} dy &= \int \left( \int |K_1(z, y) K_2(x, y)|^p dz \right)^{p'/p} dy \\ &\leq \left( \int \left( \int |K_1(z, y) K_2(x, y)|^{p'} dy \right)^{p/p'} dz \right)^{p'/p} \\ &= J^{p'}. \end{aligned}$$

Thus (2.3) gives

$$|(Tu, v)| \leq J \|u\|_p \|v\|_{p'}, \quad (2.4)$$

which gives the desired result. Note that all integrations with respect to  $y$  need only be taken over the support of  $\varphi$ . ■

Next let  $G_{s,c}(x)$  be the Green's function for the operator  $(c^2 - \Delta)^{-s/2}$ . Thus

$$G_{s,c}(x) = (4\pi)^{-n/2} \Gamma(s/2)^{-1} \int_0^\infty \exp \left\{ -\frac{|x|^2}{4t} - c^2 t \right\} t^{(s/2)-(n/2)-1} dt, \quad (2.5)$$

for  $c > 0$ . It satisfies

$$G_{s,c}(x) = c^{n-s} G_{s,1}(cx) \quad (2.6)$$

and

$$G_{s,c} * G_{t,c} = G_{s+t,c}, \quad s, t > 0. \quad (2.7)$$

Put

$$\begin{aligned} \omega_s(x) &= |x|^{s-n}, & s < n, \\ &= |\log |x||, & s = n, \\ &= 1, & s > n. \end{aligned}$$

Then there are positive constants  $a, A$  depending only on  $s$  such that

$$\omega_s(x) e^{-ac|x|}/A \leq G_{s,c}(x) \leq A\omega_s(x) e^{-ac|x|}. \quad (2.8)$$

All of these properties can be found in [8]. Note also that

$$G_{s,c}(x)^t \leq \text{const } G_{(s-n)t+n,ct}(x), \quad s \neq n. \quad (2.9)$$

Our next important statement is

**THEOREM 2.2.** *If  $1 \leq p \leq 2$ ,  $s > 0$  and  $q \in M_{sp,p}$ , then*

$$T = q(c^2 - \Delta)^{-s/2}$$

*is a bounded operator on  $L^p$  with norm bounded by*

$$C(1 + (c\delta)^{-n})^{1/p} M_{sp,p,\delta}(q)^{1/p}, \quad (2.10)$$

*where the constant  $C$  depends only on  $s$  and  $p$ .*

*Proof.* Put

$$K_1(x, y) = |q(x)| G_{s,c}(x - y)^{1/2} \quad \text{and} \quad K_2(x, y) = G_{s,c}(x - y)^{1/2}$$

in Theorem 2.1, and assume  $p > 1$ . Then

$$\begin{aligned} W(x, z) &= \int |K_1(z, y) K_2(x, y)|^{p'} dy \\ &\leq |q(z)|^{p'} \int G_{s,c}(z-y)^{p'/2} G_{s,c}(x-y)^{p'/2} dy. \end{aligned}$$

If  $s \neq n$ , we have

$$G_{s,c}^{p'/2} * G_{s,c}^{p'/2} \leq C G_{(s-n)p'+2n, cp'/2}.$$

Thus if  $sp \neq n$ ,

$$W(x, z)^{p-1} \leq C |q(z)|^p G_{sp, cp/2}(z-x). \quad (2.11)$$

Thus by Theorem 2.1 the norm of the operator  $T$  on  $L^p$  is bounded by

$$C \sup_x \left( \int |q(y)|^p G_{sp, cp/2}(x-y) dy \right)^{1/p}, \quad (2.12)$$

provided  $s \neq n$  and  $sp \neq n$ . Now for  $d > 0$ ,

$$\begin{aligned} &\int |q(y)|^p \omega_\alpha(x-y) e^{-d|x-y|} dy \\ &\leq M_{\alpha, p, \delta}(q) + \sum_{k=1}^{\infty} e^{-kd} \int_{k\delta < |x-y| < (k+1)\delta} |q(y)|^p \omega_\alpha(x-y) dy. \end{aligned} \quad (2.13)$$

Now the number of balls of radius  $\delta$  needed to cover the shell  $k\delta < |z| < (k+1)\delta$  is bounded by a constant times  $k^{p-1}$ . Thus the right-hand side of (2.13) is bounded by

$$M_{\alpha, p, \delta}(q) \left( 1 + C \sum_{k=1}^{\infty} k^{n-1} e^{-kd} \right). \quad (2.14)$$

This implies (2.10). Thus we have proved the theorem for  $s \neq n$  and  $sp \neq n$ . However, if  $s = n$ , then  $sp > n$  and the theorem follows from [2, Chap. 6, Theorem 2.1]. If  $sp = n$ , then (2.11) should be replaced by

$$W(x, z)^{p-1} \leq C |q(z)|^p |\log |z-x||^{p-1} e^{-cp/2|z-x|},$$

from which the theorem follows as before. Turning to the case  $p = 1$ , we note that the bound (2.12) follows from a trivial estimate even for the case  $s = n$ . Thus the bounded (2.10) follows from (2.14) as before. ■

We use Theorem 2.2 in proving

THEOREM 2.3. *Put*

$$M = \sum a_{jk} D_j D_k. \quad (2.15)$$

*If  $q_-$  satisfies hypothesis (C) of Theorem 1.1 for some  $p$ , then  $q_-(c^2 + M)^{-1}$  is a bounded operator on  $L^p$  and its norm tends to 0 as  $c \rightarrow \infty$ .*

*Proof.* Then operator  $(c^2 + M)^{-1}$  has a kernel  $g_c(x)$  satisfying

$$0 \leq g_c(x) \leq B |x|^{2-n} e^{-cd|x|}, \quad (2.16)$$

where the constants  $B$  and  $d$  are positive and independent of  $c$  (see, e.g., Miranda [1]). In particular, by (2.8),

$$g_c(x) \leq CG_{2,cd/a}(x).$$

Thus

$$|q_-(c^2 + M)^{-1} v| \leq C |q_-| G_{2,cd/a} * |v|,$$

and consequently by Theorem 2.2,

$$\|q_-(c^2 + M)^{-1} v\|_p^p \leq C(1 + (1/\delta cd)^n) M_{2p,p,\delta}(q_-).$$

By (1.3) this tends to 0 as  $c \rightarrow \infty$ . ■

THEOREM 2.4. *If*

$$K^2 = \iint |q(x) q(y)|^2 \omega_{2s}(x - y)^2 e^{-2c|x-y|} dx dy < \infty,$$

*then  $q(c^2 - \Delta)^{-s/2}$  is a bounded operator on  $L^2$  with bound  $\leq BK$ .*

*Proof.* We have

$$\|G_{s,c} * (qu)\|^2 = (G_{2s,c} * (qu), qu) = \iint G_{2s,c}(x - y) q(y) u(y) \overline{q(x) u(x)} dx dy.$$

The square of this equation is bounded by

$$\iint G_{2s,c}(x - y)^2 |q(x) q(y)|^2 dx dy \|u\|^2 \|v\|^2.$$

COROLLARY 2.5. *If*

$$\iint |q_-(x) q_-(y)|^2 \omega_d(x - y)^2 e^{-c|x-y|} dx dy \rightarrow 0 \quad \text{as} \quad c \rightarrow \infty, \quad (2.17)$$

*then  $|q_-|(M + c^2)^{-1}$  is a bounded operator on  $L^2$  with bound tending to 0 as  $c \rightarrow \infty$ .*

### 3. KATO'S INEQUALITY

In proving Theorem 1.1 we shall use a slight extension of an inequality due to Kato [4]. Put

$$\begin{aligned} L_0 u &= \sum a_{jk}(D_j + b_j)(D_k + b_k)u, \\ L_1 u &= Mu + 2 \sum a_{jk} D_j(b_k u) + (b - e)u, \\ L_2 u &= Mu + 2 \sum a_{jk} b_k D_j u + (b + e)u. \end{aligned}$$

It is obvious that  $L$  maps  $D$  into  $D'$ . However, this is not enough. We shall show that under hypotheses (A) and (B) it actually maps  $D$  into  $L^2$ . Note also that  $L_1$  maps  $L^2_{\text{loc}}$  into  $D'$  while  $L_2$  maps  $D$  into  $L^2$  (it also maps  $C^\infty$  into  $L^2_{\text{loc}}$ ). Before proving the inequality, we shall need a few lemmas. We shall denote the Friedrichs mollifier by  $J_\epsilon$ .

**LEMMA 3.1.** *If  $w = (I - \Delta)g$  is in  $L^1$  and  $h \in M_{1,1}$  then  $hD_k J_\epsilon g$  converges to  $hD_k g$  in  $L^1$  for each  $k$ .*

*Proof.* Note that  $D_k g = D_k G_{2,1} * w$ . It is known (see [8]) that

$$|D_k G_{2,1}(x)| \leq C G_{1,1}(x). \quad (3.1)$$

Thus

$$|hD_k(J_\epsilon - 1)g| \leq C |h| G_{1,1} * |(J_\epsilon - 1)w|,$$

and consequently

$$\|hD_k(J_\epsilon - 1)g\|_1 \leq C \|(J_\epsilon - 1)w\|_1 \rightarrow 0,$$

by Theorem 2.2. ■

**LEMMA 3.2.** *Assume  $u \in L^2_{\text{loc}}$ ,  $(I - \Delta)g \in L^1$ ,  $\text{grad } h \in L^{4/3}$  and  $u = g + h$  on some open set  $\Omega$ . Then*

$$\sum a_{jk} D_j(b_k u) = \sum a_{jk} b_k D_j u + eu \quad (3.2)$$

*holds in  $\Omega$ .*

*Proof.* We first note that it is true if  $u$  is in  $C^\infty$ . Thus (3.2) is true if we replace  $u$  by  $J_\epsilon u$ . Now by Lemma 3.1,  $b_k D_j J_\epsilon g$  converges to  $b_k D_j g$  in  $L_{\text{loc}}$ . Since  $D_j h$  is in  $L^{4/3}$ ,  $b_k D_j J_\epsilon h$  converges to  $b_k D_j h$  also in  $L_{\text{loc}}$ . Also  $D_j(b_k J_\epsilon u)$  and  $e J_\epsilon u$  converge in the sense of distributions to  $D_j(b_k u)$  and  $eu$ , respectively. Thus (3.2) holds in the limit. ■

COROLLARY 3.3. *If  $u \in C^\infty$ , then  $L_0 u = L_1 u = L_2 u$ . Thus  $L$  maps  $D$  into  $L^2$ .*

LEMMA 3.4. *If  $u \in L^2_{\text{loc}}$  and*

$$L_1 J_\epsilon u \rightarrow L_1 u \quad \text{in } L^1_{\text{loc}}, \quad (3.3)$$

*then*

$$M |u| \leq \text{Re}(\text{sgn } \bar{u}) L_1 u, \quad (3.4)$$

*where  $\text{sgn } z = z/|z|$  for  $z \neq 0$ ,  $\text{sgn } 0 = 0$ .*

*Proof.* First assume that  $u \in C^\infty$ , and put  $v^2 = u\bar{u} + \delta$ , where  $\delta > 0$ . Then

$$v D_k v = i \text{Im } \bar{u}(D_k + b_k)u. \quad (3.5)$$

Thus

$$\begin{aligned} v^2 \sum a_{jk} D_j v \overline{D_k v} &= \sum a_{jk} (\text{Im } \bar{u}(D_j + b_j)u) (\text{Im } u(D_k + b_k)u) \\ &\leq |u|^2 \sum a_{jk} (D_j + b_j)u \overline{(D_k + b_k)u}. \end{aligned} \quad (3.6)$$

Differentiating (3.5), we obtain

$$\begin{aligned} D_j(v D_k v) &= \text{Re } D_j(\bar{u}(D_k + b_k)u) \\ &= \text{Re}(\bar{u}(D_j + b_j)(D_k + b_k)u - \overline{(D_j + b_j)u} (D_k + b_k)u). \end{aligned}$$

Thus

$$v M v + \sum a_{jk} D_j v D_k v = \text{Re } \bar{u} L_0 u - \sum a_{jk} (D_j + b_j)u \overline{(D_k + b_k)u}.$$

Consequently, by (3.6),

$$M v \leq \text{Re}(\bar{u}/v) L_1 u. \quad (3.7)$$

Now suppose  $u$  satisfies only the hypotheses of the lemma, and put  $\tilde{v}^2 = |J_\epsilon u|^2 + \delta$ . Then by (3.7),

$$M \tilde{v} \leq \text{Re}(\overline{J_\epsilon u}/\tilde{v}) L_1 J_\epsilon u. \quad (3.8)$$

By considering a subsequence, we can make  $\overline{J_\epsilon u}/\tilde{v}$  converge to  $\bar{u}/v$  a.e. as  $\epsilon \rightarrow 0$ . Since it is uniformly bounded, we see by (3.3) that the right-hand side of (3.8) converges to the right-hand side of (3.7) in  $L^1_{\text{loc}}$ . Moreover, the left-hand side of (3.8) converges to the left-hand side of (3.7) in the sense of distributions. Thus (3.7) holds for such  $u$ . We now let  $\delta \rightarrow 0$ . ■



LEMMA 3.5. *If the hypotheses of Lemma 3.2 hold and  $Mu$  is in  $L_{10c}$ , then (3.3) holds on  $\Omega$ .*

*Proof.* By Lemma 3.2,  $L_1u = L_2u$ . Therefore, it suffices to show that  $L_2J_\epsilon u \rightarrow L_2u$ . Since  $Mu$  is in  $M_{10c}$ ,  $MJ_\epsilon u = J_\epsilon Mu$  converges to  $Mu$  in  $L_{10c}$ . The other terms converge as in the proof of Lemma 3.2. ■

THEOREM 3.6. *If  $u \in L_{10c}^2$  and  $L_1u \in L_{10c}$ , then (3.4) holds.*

*Proof.* Put  $w = L_1u$ , and let  $\Omega$  be any bounded open set in  $E^n$ . Let  $\varphi$  be a test function which equals 1 on  $\Omega$ . Put

$$g = (I + M)^{-1}(w + u - eu)\varphi, \quad h = -2(I + M)^{-1} \sum a_{jk} D_j(b_k \varphi u).$$

Since  $(I + M)g \in L^1$ , the same is true of  $(I - \Delta)g$ . Moreover,  $\text{grad } h \in L^{4/3}$ . Also,  $(I + M)(g + h - u) = 0$  in  $\Omega$ , and consequently  $g + h - u \in C^\infty$  there. Thus the hypotheses of Lemma 3.2 are satisfied. From this and Lemma 3.1 we see that  $(I + M)h \in L^1$ , and thus  $Mu \in L_{10c}$ . Apply Lemma 3.5. ■

#### 4. PROOF

In this section we give the proof of Theorem 1.1. We do this by showing that for  $c$  sufficiently large the operator  $L + c^2$  on  $D$  has range dense in  $L^2$  (note that it maps  $D$  into  $L^2$  by Corollary 3.3). If this were not so, there would be a  $u \in L^2$  such that  $u \neq 0$  and

$$(u, (L + c^2)\varphi) = 0, \quad \varphi \in D.$$

This gives

$$(L_1 + q + c^2)u = 0, \tag{4.1}$$

in the sense of distributions. This implies that  $L_1u$  is in  $L_{10c}$ . Thus (3.4) holds in view of Theorem 3.6. Hence

$$M|u| \leq \text{Re}(\text{sgn } \bar{u})(-q - c^2)u \leq -(q_- + c^2)|u|.$$

Hence

$$(M + c^2)|u| \leq |q_- u|. \tag{4.2}$$

Before proceeding further, we state two simple lemmas.

LEMMA 4.1. *If  $\varphi \in D$  and  $cd \geq s$ , then  $e^{s|x|}(M + c^2)^{-1}\varphi$  is bounded.*

*Proof.* We have

$$\begin{aligned} |(M + c^2)^{-1} \varphi| &\leq \int g_c(x - y) |\varphi(y)| dy \\ &\leq B \int |x - y|^{2-n} e^{-cd|x-y|} |\varphi(y)| dy \\ &\leq B e^{-cd|x|} \int |x - y|^{2-n} e^{cd|y|} |\varphi(y)| dy. \end{aligned}$$

Since  $\varphi \in D$ , the integral is uniformly bounded. ■

LEMMA 4.2. *If  $v(x) = e^{-s|x|} w(x)$  is in  $L^1$  and  $cd \geq s$ , then*

$$(M + c^2)^{-1} w \leq B e^{s|x|} (M + (c - t)^2)^{-1} |v|,$$

where  $t = s/d$ .

*Proof.* We have

$$\begin{aligned} \int g_c(x - y) |w(y)| dy &\leq B \int |x - y|^{2-n} e^{-cd|x-y|} |w(y)| dy \\ &\leq B e^{s|x|} \int |x - y|^{2-n} e^{(s-cd)|x-y|} |v(y)| dy \\ &\leq B^2 e^{s|x|} (M + (c - t)^2)^{-1} |v|. \quad \blacksquare \end{aligned}$$

Returning to our proof, first consider the case  $p < 2$  in assumption (C). Note that

$$(\psi, (M + c^2) |u|) \leq (\psi, |q_- u|) \quad (4.3)$$

holds for any nonnegative  $\psi$  in  $S$  (the Schwartz space of rapidly decreasing functions) such that  $e^{l|x|}\psi$  is bounded. This follows from the fact that  $v(x) = e^{-l|x|} |q_-(x) u(x)|$  is in  $L^p$  by assumption (D). In view of Lemma 4.1 we may take  $\psi = (M + c^2)^{-1}\varphi$ , where  $\varphi \geq 0$  is in  $D$ . This implies

$$|u| \leq (M + c^2)^{-1} |q_- u|, \quad (4.4)$$

in the sense of distributions. By Lemma 4.2 this implies

$$v \leq B |q_-| (M + c_1^2)^{-1} v, \quad (4.5)$$

where  $c_1 = c - (l/d)$ . In view of Theorem 2.3, assumption (C) implies that the norm of the operator  $|q_-| (M + c_1^2)^{-1}$  on  $L^p$  tends to 0 as  $c_1 \rightarrow \infty$ . Thus for  $c$  sufficiently large, (4.5) implies

$$\|v\|_p \leq \frac{1}{2} \|v\|_p,$$

which can only happen if  $v = 0$ . This implies  $u = 0$  via (4.4).

Next consider the case  $p = 2$  in hypothesis (C). This now implies that

$$\int_{|x| < R} |q_-(x)|^2 dx \leq CR^k$$

for some integer  $k$ , and consequently that  $q_-u$  is in  $S'$ . Thus we may go immediately from (4.2) to (4.4). Thus

$$\|u\|_2 \leq \|(M + c^2)^{-1} |q_-| \| \|u\|_2.$$

But  $\|(M + c^2)^{-1} |q_-| \| = \| |q_-| (M + c^2)^{-1} \|$  on  $L^2$ , and this norm tends to 0 as  $c \rightarrow \infty$ , by Theorem 2.3. Thus  $u = 0$ , and the proof of Theorem 1.1 is complete. ■

Now we show how hypothesis (C) can be replaced by others.

**THEOREM 4.3.** *Theorem 1.1 is true if hypotheses (C) and (D) are replaced by (2.17).*

*Proof.* By (2.17),  $q_-$  is in  $L^2$ . Thus  $q_-u$  is integrable and consequently in  $S'$ . Thus we may proceed directly from (4.2) to (4.4). We now proceed as in the proof of Theorem 1.1, making use of Corollary 2.5. ■

**THEOREM 4.4.** *The same is true if (C) and (D) are replaced by (E)  $q_- \in L^{n/2}$ ,  $n > 4$ .*

*Proof.* It is standard that (E) also implies the conclusions of Corollary 2.5 (see, e.g., [4]). ■

*Remark 1.* The condition (2.17) is not usable for  $n > 7$ .

*Remark 2.* When  $n < 8$ , condition (E) implies (2.17).

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